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METHOD OF AVERAGING IN THE PROBLEMS OF STABILITY OF ELASTIC PLATES POSSESSING FINE PERIODIC STRUCTURE*

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Buckling of an elastic plate with mechanical properties inhomogeneous across its thickness is studied. The character of the inhomogeneity is described by periodic rapidly oscillating functions of the transverse coordinate. The method of averaging is used to construct an asymptotic to the solution of the problem of stability in the case when the period of the inhomogeneity oscillations tends to zero. An averaged system of differential equations is obtained in the case of a homogeneous (affine) subcritical deformation, separately, for the case of a compressible and an incompressible materials. The system is used for determining, in the given approximation, the bifurcation values of the load parameters. The general theory is illustrated by an example of computing the stability of a rectangular plate uniformly compressed in its plane, with the plate made of a resin-like incompressible one-constant material, the modulus of elasticity of which is a rapidly oscillating function of the transverse coordinate. The results obtained using the method of averaging are compared, for a thin plate, with those of the buckling theory based on the Kirchhoff hypothesis.

1. Let us consider an elastic plate bounded in the undeformed state by the planes $x_3 = \pm h$. We assume the plate material to be orthotropic (in particular isotropic), and the x_3 axis lies in one of the material symmetry planes. Elastic properties of the material are assumed homogeneous with respect to the Cartesian x_1, x_2 coordinates counted on the middle surface of the plate, and inhomogeneous along the transverse coordinate x_3. This means that the specific potential energy of deformation W, which is a function of the Cauchy–Green deformation tensor, will depend explicitly on the coordinate x_3. The dependence is assumed to be $2h\varepsilon$-periodic. The dimensionless parameter ε will be assumed small (the case of a fine periodic structure).

We assume that the elastic body in question is subjected to an initial deformation of the following form: x_3 is the principal axis of the deformation tensor, the planes $x_3 = \mathrm{const}$ experience an affine deformation independent of x_3, stresses are absent from these surfaces, and the elongation of the fibers orthogonal to the middle surface depends, in general, on x_3. It can be shown that such a state satisfies the equilibrium equation for an orthotropic material inhomogeneous in the direction of the transverse coordinate when the mass forces are zero, by virtue of the forces distributed along the side surfaces of the plate. In the case of an incompressible material in the same state of equilibrium, the elongations of the transverse fibers are independent of x_3, i.e. the deformation will be homogeneous over the whole body.

The equation of neutral equilibrium describing buckling of a plate, have the form /1/

$$\nabla\cdot\mathbf{D}^{\cdot} = 0, \quad \mathbf{D}^{\cdot} = \frac{d}{d\eta}\mathbf{D}(\mathbf{R} + \eta\mathbf{u})|_{\eta=0}, \quad \nabla = \frac{\partial}{\partial x_k}\mathbf{i}_k \tag{1.1}$$

Here $\mathbf{D}$ is the asymmetric Piola stress tensor, $\mathbf{R}$ is the radius vector of the points of the body in subcritical state, $\mathbf{u}$ is the vector of additional displacements, $\mathbf{i}_k$ $(k = 1, 2, 3)$ are the Cartesian coordinates unit vectors, and ∇ is the del operator in the undeformed configuration of the body. An upper dot denotes the perturbations (linear increments) resulting from the additional displacements.

For a compressible material we have /1,2/

$$\mathbf{D}^{\cdot} = \mathbf{K}\cdot\cdot(\nabla\mathbf{u})^T, \quad \mathbf{K} = K_{pqrs}\mathbf{i}_p\mathbf{i}_q\mathbf{i}_r\mathbf{i}_s, \quad K_{pqrs} = \frac{\partial^2 W}{\partial C_{pq}\partial C_{rs}} \tag{1.2}$$

Here C_{pq} are the subcritical state deformation gradient components, and $\mathbf{K}$ is the elasticity tensor, which in this case is a periodic function of the coordinate x_3 and independent of the

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coordinates x_1 and x_2. The boundary conditions on the face planes of the plate expressing the absence of load in the perturbed state of equilibrium, are written as follows:

$$\mathbf{i}_3 \cdot \mathbf{D}^{\cdot} = 0 \quad \text{when} \quad x_3 = \pm h \tag{1.3}$$

Using the second rank tensors

$$\mathbf{A}_{mn} = K_{mqns}\mathbf{i}_q\mathbf{i}_s \quad (m, n = 1, 2, 3)$$

we can write the equations (1.1) and boundary conditions (1.3) in the form

$$\partial_m \mathbf{A}_{mn} \cdot \partial_n \mathbf{u} = 0 \tag{1.4}$$

$$\mathbf{A}_{3n} \cdot \partial_n \mathbf{u} = 0 \quad \text{when} \quad x_3 = \pm h \tag{1.5}$$

Since the tensors $\mathbf{A}_{mn}$ are independent of x_1 and x_2, equations (1.4) and conditions (1.5) admit solutions of the form

$$\mathbf{u} = \mathbf{f}(x_3) \exp[i(\alpha x_1 + \beta x_2)] \tag{1.6}$$

which make it possible to satisfy certain boundary conditions on the side surface of the plate. The conditions which are of sufficient interest include the conditions of hinged support or sliding clamp at the edges of a rectangular plate.

Substituting (1.6) into (1.4), (1.5) we arrive at a system of ordinary differential equations for three functions $f_s = \mathbf{f} \cdot \mathbf{i}_s$ (operator d denotes differentiation with respect to x_3)

$$[d\mathbf{A}_{33}d + (i\alpha\mathbf{A}_{13} + i\beta\mathbf{A}_{23})d + d(i\alpha\mathbf{A}_{31} + i\beta\mathbf{A}_{32}) - \alpha^2\mathbf{A}_{11} - \beta^2\mathbf{A}_{22} - \alpha\beta(\mathbf{A}_{12} + \mathbf{A}_{21})] \cdot \mathbf{f} = 0 \tag{1.7}$$

with boundary conditions at $x_3 = \pm h$

$$(\mathbf{A}_{33}d + i\alpha\mathbf{A}_{31} + i\beta\mathbf{A}_{32}) \cdot \mathbf{f} = 0 \tag{1.8}$$

Setting

$$\mathbf{g} = (\mathbf{A}_{33}d + i\alpha\mathbf{A}_{31} + i\beta\mathbf{A}_{32}) \cdot \mathbf{f}$$

we obtain, in accordance with (1.7)

$$\begin{aligned} d\mathbf{f} &= -(\mathbf{A}_{33})^{-1} \cdot \mathbf{P} \cdot \mathbf{f} + (\mathbf{A}_{33})^{-1} \cdot \mathbf{g} \\ d\mathbf{g} &= (\mathbf{B} \cdot (\mathbf{A}_{33})^{-1} \cdot \mathbf{P} - \mathbf{Q}) \cdot \mathbf{f} - \mathbf{B} \cdot (\mathbf{A}_{33})^{-1} \cdot \mathbf{g} \\ \mathbf{P} &= i\alpha\mathbf{A}_{31} + i\beta\mathbf{A}_{32}; \quad \mathbf{B} = i\alpha\mathbf{A}_{13} + i\beta\mathbf{A}_{23} \\ \mathbf{Q} &= -\alpha^2\mathbf{A}_{11} - \beta^2\mathbf{A}_{22} - \alpha\beta(\mathbf{A}_{12} + \mathbf{A}_{21}) \end{aligned} \tag{1.9}$$

Using (1.9) we can write the system of equations (1.7) as follows (Λ is a matrix with tensor elements)

$$d \begin{Vmatrix} \mathbf{f} \\ \mathbf{g} \end{Vmatrix} = \Lambda \cdot \begin{Vmatrix} \mathbf{f} \\ \mathbf{g} \end{Vmatrix}, \quad \Lambda = \begin{Vmatrix} -(\mathbf{A}_{33})^{-1} \cdot \mathbf{P}, & (\mathbf{A}_{33})^{-1} \\ \mathbf{B} \cdot (\mathbf{A}_{33})^{-1} \cdot \mathbf{P} - \mathbf{Q}, & -\mathbf{B} \cdot (\mathbf{A}_{33})^{-1} \end{Vmatrix} \tag{1.10}$$

We solve the problem of stability of a plate with fine periodic structure using the method of averaging /3–6/. Setting $y = \varepsilon^{-1}x_3$, we seek the solution of (1.10) in the form of a power series

$$\mathbf{f}(x_3, y) = \sum_{k=0}^{\infty} \mathbf{f}^{(k)}(x_3, y)\varepsilon^k \tag{1.11}$$

$$\mathbf{g}(x_3, y) = \sum_{k=0}^{\infty} \mathbf{g}^{(k)}(x_3, y)\varepsilon^k$$

where $\mathbf{f}^{(k)}(x_3, y)$, $\mathbf{g}^{(k)}(x_3, y)$ are $2h$-periodic functions of y. The base unknowns in the plate buckling problem are those values of the load parameters (i.e. the parameters determining the subcritical state), for which the system of equations (1.7) with boundary conditions (1.8) has a nontrivial solution. The unknown critical values of the load parameters must also be sought in the form of a power series in terms of the dimensionless period ε. It is for this reason that the coefficients of the system (1.7) dependent on the load parameters will appear in the form of series

$$\mathbf{A}_{mn} = \sum_{k=0}^{\infty} \mathbf{A}_{mn}^{(k)}(y)\varepsilon^k \tag{1.12}$$

By virtue of the obvious relation

$$\frac{d}{dx_3}\mathbf{f}(x_3, y) = \frac{\partial \mathbf{f}}{\partial x_3} + \varepsilon^{-1}\frac{\partial \mathbf{f}}{\partial y}$$

the differentiation operator is written in the form

$$d = \partial + \varepsilon^{-1}\,\partial/\partial y,\ \partial = \partial/\partial x_3 \tag{1.13}$$

Taking (1.13) into account, we write the system (1.10) in the form

$$\varepsilon^{-1}\partial_y\left\|\begin{matrix}\mathbf{f}\\ \mathbf{g}\end{matrix}\right\| + \partial\left\|\begin{matrix}\mathbf{f}\\ \mathbf{g}\end{matrix}\right\| - \Lambda\cdot\left\|\begin{matrix}\mathbf{f}\\ \mathbf{g}\end{matrix}\right\| = 0 \tag{1.14}$$

Substituting the expansions (1.11) and (1.12) into (1.14), we equate to zero, one after the other, the coefficients accompanying like powers of ε. Equating to zero the coefficients of ε^{-1} we obtain

$$\partial_y\mathbf{f}^{(0)} = 0,\ \partial_y\mathbf{g}^{(0)} = 0$$

and from this follows $\mathbf{f}^{(0)}(x_3, y) = \mathbf{f}^{(0)}(x_3),\ \mathbf{g}^{(0)}(x_3, y) = \mathbf{g}^{(0)}(x_3)$. This means that the principal term of the expansion (1.11) is not a rapidly oscillating function, but represents a component of the solution of the stability problem, varying slowly across the thickness of the plate. The coefficient of ε^0 yields the equations

$$\partial_y\left\|\begin{matrix}\mathbf{f}^{(1)}\\ \mathbf{g}^{(1)}\end{matrix}\right\| + \partial\left\|\begin{matrix}\mathbf{f}^{(0)}\\ \mathbf{g}^{(0)}\end{matrix}\right\| = \Lambda^{(0)}\cdot\left\|\begin{matrix}\mathbf{f}^{(0)}\\ \mathbf{g}^{(0)}\end{matrix}\right\| \tag{1.15}$$

Averaging the system (1.15), we arrive at a system of equations in $f^{(0)},\ g^{(0)}$

$$\partial\left\|\begin{matrix}\mathbf{f}^{(0)}\\ \mathbf{g}^{(0)}\end{matrix}\right\| = \langle\Lambda^{(0)}\rangle\cdot\left\|\begin{matrix}\mathbf{f}^{(0)}\\ \mathbf{g}^{(0)}\end{matrix}\right\| \tag{1.16}$$

$$\langle\Lambda^{(0)}\rangle = \left\|\begin{matrix}-\langle(\mathbf{A}_{33}^{(0)})^{-1}\cdot\mathbf{P}^{(0)}\rangle, & \langle(\mathbf{A}_{33}^{(0)})^{-1}\rangle\\ \langle\mathbf{B}^{(0)}\cdot(\mathbf{A}_{33}^{(0)})^{-1}\cdot\mathbf{P}^{(0)} - \mathbf{Q}^{(0)}\rangle, & -\langle\mathbf{B}^{(0)}\cdot(\mathbf{A}_{33}^{(0)})^{-1}\rangle\end{matrix}\right\|$$

$$\langle\varphi(x_3, y\rangle = \frac{1}{2h}\int_{-h}^{h}\varphi(x_3, y)\,dy$$

Here and henceforth the angle brackets will denote averaging over y. The system (1.16) can be written in the form ($\mathbf{E}$ is a unit tensor)

$$\partial\left\|\begin{matrix}\mathbf{f}^{(0)}\\ \mathbf{h}^{(0)}\end{matrix}\right\| = \left\|\begin{matrix}\mathbf{0} & \mathbf{E}\\ \mathbf{V} & \mathbf{W}\end{matrix}\right\|\cdot\left\|\begin{matrix}\mathbf{f}^{(0)}\\ \mathbf{h}^{(0)}\end{matrix}\right\| \tag{1.17}$$

$$\mathbf{h}^{(0)} = -\langle(\mathbf{A}_{33}^{(0)})^{-1}\cdot\mathbf{P}^{(0)}\rangle\cdot\mathbf{f}^{(0)} + \langle(\mathbf{A}_{33}^{(0)})^{-1}\rangle\cdot\mathbf{g}$$

$$\mathbf{V} = \langle(\mathbf{A}_{33}^{(0)})^{-1}\rangle\cdot(\langle\mathbf{B}^{(0)}\cdot(\mathbf{A}_{33}^{(0)})^{-1}\cdot\mathbf{P}^{(0)}\rangle - \langle\mathbf{Q}^{(0)}\rangle - \langle\mathbf{B}^{(0)}\cdot(\mathbf{A}_{33}^{(0)})^{-1}\rangle\cdot\langle(\mathbf{A}_{33}^{(0)})^{-1}\rangle^{-1}\cdot\langle(\mathbf{A}_{33}^{(0)})^{-1}\cdot\mathbf{P}^{(0)}\rangle)$$

$$\mathbf{W} = -\langle(\mathbf{A}_{33}^{(0)})^{-1}\cdot\mathbf{P}^{(0)}\rangle - \langle(\mathbf{A}_{33}^{(0)})^{-1}\rangle\cdot\langle\mathbf{B}^{(0)}\cdot(\mathbf{A}_{33}^{(0)})^{-1}\rangle\,\langle(\mathbf{A}_{33}^{(0)})^{-1}\rangle^{-1}$$

Eliminating from (1.17) the vector $\mathbf{h}^{(0)}$, we obtain the following expression for the vector $\mathbf{f}^{(0)}$:

$$\partial^2\mathbf{f}^{(0)} - \mathbf{W}\cdot\partial\mathbf{f}^{(0)} - \mathbf{V}\cdot\mathbf{f}^{(0)} = 0 \tag{1.18}$$

The boundary conditions for $\mathbf{f}^{(0)}$ are obtained from (1.8) and have the form

$$(\mathbf{A}_{33}^{(0)}\partial + i\alpha\mathbf{A}_{31}^{(0)} + i\beta\mathbf{A}_{32}^{(0)})\cdot\mathbf{f}^{(0)} = 0 \quad \text{when} \quad x_3 = \pm h \tag{1.19}$$

The averaged system of equations (1.18) and boundary conditions (1.19) are used to obtain the principal term of the asymptotics, as $\varepsilon \to 0$, of the critical load, and of the form of elastic plate loss of stability. The approximation $\mathbf{f}^{(0)}$ in question is the more accurate, the finer the structure of the inhomogeneity, i.e. the larger the number of the inhomogeneity periods accommodated within the plate thickness.

2. In the case of an incompressible material the linearized Piola stress tensor has the form

$$\mathbf{D}^{\cdot} = \mathbf{K}\cdot\cdot(\nabla\mathbf{u})^T + r(\mathbf{C}^{-1})^T \tag{2.1}$$

where $\mathbf{C}$ is the deformation gradient of the initial deformed state of the plate, and r is an unknown function of the coordinates the supplementary equation for which is represented, together with the equations of equilibrium (1.1), by the linearized condition of incompressibility /1/

$$\mathbf{C}^{-1}\cdot\cdot\nabla\mathbf{u}=S_{mn}\frac{\partial u_m}{\partial x_n}=0;\quad S_{mn}=\mathbf{i}_m\cdot\mathbf{C}^{-1}\cdot\mathbf{i}_n \tag{2.2}$$

Substituting (2.1) into (1.1) we obtain

$$\frac{\partial}{\partial x_m}\left(K_{mnts}\frac{\partial u_s}{\partial x_t}\right)+\frac{\partial r}{\partial x_m}S_{nm}=0\quad (n=1,2,3) \tag{2.3}$$

where u_s are the components of the displacement vector and S_{mn} are the components of the tensor $\mathbf{C}^{-1}$. Equations (2.2) and (2.3), together form a system for determining the unknown functions u_m and r. To obtain the boundary conditions, we substitute (2.1) into (1.3).

$$\left(K_{3nts}\frac{\partial u_s}{\partial x_t}+rS_{n3}\right)\mathbf{i}_n=0\quad\text{when}\quad x_3=\pm h \tag{2.4}$$

Setting $r=\partial q/\partial x_3$ and differentiating (2.2) with respect to x_3 we obtain, in place of (2.2), (2.3), the system

$$\partial_m A_{mn\tau\varkappa}\partial_n v_\varkappa=0 \tag{2.5}$$
$$A_{mn\tau\varkappa}=K_{mn\tau\varkappa},\ v_\varkappa=u_\varkappa,\quad \text{if}\ \varkappa\leqslant 3,\ \tau\leqslant 3$$
$$A_{mn44}=A_{mnt4}=A_{mn4t}=0,\ v_4=q,\quad \text{if}\ m\neq 3$$
$$A_{3n4t}=A_{3nt4}=S_{tn},\ A_{3n44}=0$$

where the Latin indices assume values from 1 to 3 and the Greek indices the values from 1 to 4. We supplement the boundary conditions (2.4) with the relations

$$S_{mn}\partial_n u_m=0\ \text{when}\ x_3=\pm h \tag{2.6}$$

derived from (2.2), and write the conditions (2.4), (2.6) in the form

$$A_{3n\tau\varkappa}\partial_n v_\varkappa=0\ \text{when}\ x_3=\pm h \tag{2.7}$$

We see from (2.5), (2.7) that the system of equations of neutral equilibrium and boundary conditions for an incompressible material is represented in the form analogous to (1.4), (1.5). This implies that we can use the algorithm described in Sect.1. As a result, we obtain the averaged system of differential equations for the components of the four-dimensional vector, and the corresponding boundary conditions.

3. As an example we consider the problem of bifurcation of equilibrium in a plate uniformly compressed in its plane and made of a highly elastic, isotropic incompressible Bartenev — Khazanovich material /7/. The deformation gradient of the subcritical state has the form

$$\mathbf{C}=\lambda(\mathbf{i}_1\mathbf{i}_1+\mathbf{i}_2\mathbf{i}_2)+\lambda^{-2}\mathbf{i}_3\mathbf{i}_3;\quad \lambda=\text{const} \tag{3.1}$$

The Cauchy stress tensor for the given material is defined by the relation /1/

$$\mathbf{T}=2\mu\mathbf{F}^{1/2}-\sigma\mathbf{E};\quad \mathbf{F}=\mathbf{C}^T\cdot\mathbf{C} \tag{3.2}$$

Here $\mathbf{F}$ is the Finger measure of deformation, $\mu=\mu(\varepsilon^{-1}x_3)$ is the modulus of elasticity of the material, representing a rapidly oscillating function of the transverse coordinate, and σ is the pressure in the incompressible body which cannot be determined by the deformation. In the subcritical state the quantity σ is found from the condition of absence of the normal transverse stress, and has the following values:

$$\sigma=2\mu\lambda^{-2} \tag{3.3}$$

Taking due account of the known /1/ relation $\mathbf{C}^T\cdot\mathbf{D}=(\det\mathbf{C})\,\mathbf{T}$, we obtain

$$\mathbf{D}^{\cdot}=(\mathbf{C}^{-1})^T\cdot\mathbf{T}^{\cdot}-(\mathbf{C}^{-1})^T\cdot(\nabla\mathbf{u})^T\cdot(\mathbf{C}^{-1})^T\cdot\mathbf{T} \tag{3.4}$$

Using now a formula given in /8/ for a derivative of the square root of a tensor, from (3.2) we find ($p=r/2\mu$)

$$\begin{aligned}\mathbf{T}^{\cdot}=\;&\mu(I_1I_2-1)^{-1}[I_1I_2\mathbf{F}^{\cdot}-(\mathbf{F}^{-1/2}\cdot\mathbf{F}^{\cdot}+\mathbf{F}^{\cdot}\cdot\mathbf{F}^{-1/2})+\\&(I_1^2+I_2)\mathbf{F}^{-1/2}\cdot\mathbf{F}^{\cdot}\cdot\mathbf{F}^{-1/2}-I_1(\mathbf{F}^{-1/2}\cdot\mathbf{F}^{\cdot}\cdot\mathbf{F}^{-1}+\mathbf{F}^{-1}\cdot\mathbf{F}^{\cdot}\cdot\mathbf{F}^{-1/2})+\\&\mathbf{F}^{-1}\cdot\mathbf{F}^{\cdot}\cdot\mathbf{F}^{-1}]+2\mu p\mathbf{E}\\\mathbf{F}^{\cdot}=\;&(\nabla\mathbf{u})^T\cdot\mathbf{C}+\mathbf{C}^T\cdot\nabla\mathbf{u}\\I_1=\;&2\lambda^{-1}+\lambda^2,\quad I_2=2\lambda+\lambda^{-2}\end{aligned} \tag{3.5}$$

and from (1.1), (3.1) — (3.5) we obtain the following system of equations of neutral equilibrium:

$$\mu\left(\partial_3^2u_l+\omega^2(\partial_1^2u_l+\partial_2^2u_l)+\omega^2\partial_l p+\frac{1-\gamma^2}{2\gamma^2}\partial_l\partial_3u_3\right)+(\partial_3u_l+\gamma^2\partial_lu_3)\partial_3\mu=0 \tag{3.6}$$

$$\mu\left(\gamma^2\left(\partial_1^2 u_3+\partial_2^2 u_3\right)+2\omega^2 \partial_3^2 u_3+\omega^2 \partial_3 p+\gamma^4 \partial_3\left(\partial_1 u_1+\partial_2 u_2\right)\right)+\omega^2\left(2 \partial_3 u_3+p\right) \partial_3 \mu=0$$

$$\gamma=\lambda^{-3/2};\ 2\omega^2=1+\gamma^2\ (l=1,2)$$

The condition of incompressibility has the form

$$\gamma^2\left(\partial_1 u_1+\partial_2 u_2\right)+\partial_3 u_3=0 \tag{3.7}$$

and the following boundary conditions must hold at the end faces of the plate:

$$\partial_3 u_l+\gamma^2 \partial_l u_3=0,\quad 2 \partial_3 u_3+p=0 \text{ for } x_3=\pm h\quad (l=1,2) \tag{3.8}$$

Setting

$$u_k=f_k\left(x_3\right) \exp \left[i\left(\alpha x_1+\beta x_2\right)\right]\ (k=1,2,3),\quad p=f_4\left(x_3\right) \exp \left[i\left(\alpha x_1+\beta x_2\right)\right]$$

and averaging, we obtain

$$\begin{aligned}
&\partial^2 f_1^{(0)}-\nu \zeta^2\left(\omega^{(0)}\right)^2 f_1^{(0)}+i \alpha c \partial f_3^{(0)}+i \alpha\left(\omega^{(0)}\right)^2 f_4^{(0)}=0 \\
&\partial^2 f_2^{(0)}-\nu \zeta^2\left(\omega^{(0)}\right)^2 f_2^{(0)}+i \beta c \partial f_3^{(0)}+i \beta\left(\omega^{(0)}\right)^2 f_4^{(0)}=0 \\
&\partial^2 f_3^{(0)}-z \zeta^2\left(z^2-\nu\left(z^2-1\right)\right) f_3^{(0)}+\left(\omega^{(0)}\right)^2 \partial f_4^{(0)}=0 \\
&z\left(i \alpha f_1^{(0)}+i \beta f_2^{(0)}\right)+\partial f_3^{(0)}=0 \\
&\zeta=\sqrt{\alpha^2+\beta^2},\quad \gamma^{(0)}=\left(\lambda^{(0)}\right)^{-3/2},\quad z=\left(\gamma^{(0)}\right)^2, \\
&2\left(\omega^{(0)}\right)^2=1+z,\quad c=1+2 z+\nu \frac{1-3 z-4 z^2}{2 z}
\end{aligned} \tag{3.9}$$

Here $\lambda^{(0)}$ is the first term of the series $\lambda=\lambda^{(0)}+\lambda^{(1)}\varepsilon+\lambda^{(2)}\varepsilon^2+\ldots$, and $\nu=\langle\mu\rangle\langle\mu^{-1}\rangle$ is a parameter characterizing the inhomogeneity of the plate. For the homogeneous plate we have $\nu=1$. The general solution of the system (3.9) has the form

$$\begin{aligned}
&f_1^{(0)}=\alpha\left(C_1 E_{13}^{+}+C_2 E_{23}^{+}+C_3 E_{13}^{-}+C_4 E_{23}^{-}\right)+\alpha^{-1}\left(C_5 E_{33}^{+}+C_6 E_{33}^{-}\right) \\
&f_2^{(0)}=\beta\left(C_1 E_{13}^{+}+C_2 E_{23}^{+}+C_3 E_{13}^{-}+C_4 E_{23}^{-}\right)-\beta^{-1}\left(C_5 E_{33}^{+}+C_6 E_{33}^{-}\right) \\
&f_3^{(0)}=-i \zeta^2 z\left(\frac{C_1}{m_1} E_{13}^{+}+\frac{C_2}{m_2} E_{23}^{+}-\frac{C_3}{m_1} E_{13}^{-}-\frac{C_4}{m_2} E_{23}^{-}\right) \\
&f_4^{(0)}=d_1\left(C_1 E_{13}^{+}+C_3 E_{13}^{-}\right)+d_2\left(C_2 E_{23}^{+}+C_4 E_{23}^{-}\right) \\
&E_{ij}^{\pm}=\exp \left(\pm m_i x_j\right);\quad j, i=1,2,3;\quad M=\left(1-\nu^{-1}\right)^{1/2}; \\
&m_1=\gamma^{(0)} \zeta\left[-z+\nu(1+z)(1+M)\right]^{1/2};\quad m_2=\gamma^{(0)} \zeta\left[-z+\nu(1+z)(1-M)\right]^{1/2};\quad m_3=\zeta \omega^{(0)} \nu^{1/2} \\
&d_l=i \zeta^2 z\left[m_l^2-z \zeta^2\left(z^2-\nu\left(z^2-1\right)\right)\right]\left(\omega^{(0)}\right)^{-2} m_l^{-2}\ (l=1,2)
\end{aligned} \tag{3.10}$$

When $x_3=\pm h$, the following conditions must hold:

$$\partial f_1^{(0)}+i \alpha z f_3^{(0)}=0,\quad \partial f_2^{(0)}+i \beta z f_3^{(0)}=0,\quad 2 \partial f_3^{(0)}+f_4^{(0)}=0 \tag{3.11}$$

Substituting (3.10) into (3.11) we obtain the equation for finding $\lambda^{(0)}$.

The boundary value problem (3.9), (3.11) can be separated into two, mutually independent problems.

Problem A: $f_1^{(0)}, f_2^{(0)}, f_4^{(0)}$ are even functions and $f_3^{(0)}$ is an odd function of the coordinate x_3.

Problem B: $f_1^{(0)}, f_2^{(0)}, f_4^{(0)}$ are odd functions and $f_3^{(0)}$ is an even function of the coordinate x_3.

Problem A describes the symmetric, and B the antisymmetric, i.e. flexural forms of the loss of stability of the plate. The equation for determining $\lambda^{(0)}$ in Problem A has the form

$$m_1(1+M)(1-z M) \operatorname{th}\left(m_1 h\right)-m_2(1-M)(1+z M) \operatorname{th}\left(m_2 h\right)=0 \tag{3.12}$$

and in Problem B

$$m_1(1+M)(1-z M) \operatorname{th}\left(m_2 h\right)-m_2(1-M)(1+z M) \times \operatorname{th}\left(m_1 h\right)=0 \tag{3.13}$$

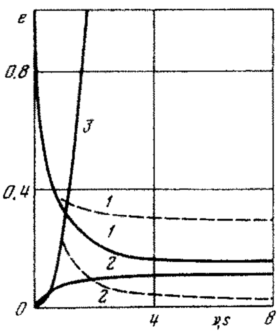
For a homogeneous plate, i.e. when $\nu=1$, we have ($s=\zeta h$ denotes the relative plate thickness)

$$\left(\gamma^2-3\right) \operatorname{sh}(2 \gamma s)=2 \gamma s\left(1+\gamma^2\right) \text{ in problem A} \tag{3.14}$$

$$\left(3-\gamma^2\right) \operatorname{sh}(2 \gamma s)=2 \gamma s\left(1+\gamma^2\right) \text{ in problem B} \tag{3.15}$$

The roots of (3.12)—(3.15) were found by numerical methods. The results show that the solutions of (3.12) and (3.13) tend monotonously to unity with increasing parameter ν, beginning with the solutions of (3.14) and (3.15), respectively. At finite values of ν and at small or finite values of s the flexural forms of the loss of stability appear before the symmetric forms. When ν tends to infinity, i.e. in the case of a strongly inhomogeneous plate, the solutions of (3.12) and (3.13) coincide asymptotically and we have

$$\lambda^{(0)} = 1 - \frac{1}{6}\nu^{-1} + o(\nu^{-1})$$



The dashed lines in the figure depict the dependence of the critical deformation $e = 1 - \lambda^{(0)}$ on the values of the parameter ν, for $s = 1$. The curves *1* and *2* are constructed for Problem A and B, respectively. It can be shown that if the functions

$$\mathbf{u} = f(x_3)\exp[i(\alpha x_1 + \beta x_2)],\quad p = f_4(x_3)\exp[i(\alpha x_1 + \beta x_2)]$$

satisfy the system (3.6), (3.7), then the expressions

$$\mathbf{u} = f(x_3)\exp[i(\pm\alpha x_1 \pm \beta x_2],\quad p = f_4(x_3)\exp[i(\pm\alpha x_1 \pm \beta x_2)]$$

and any linear combination of these expressions will also be a solution of the system in question. In particular, the following expressions will be the solutions:

$$u_1 = f_1(x_3)\cos\alpha x_1 \sin\beta x_2,\quad u_2 = f_2(x_3)\sin\alpha x_1\cos\beta x_2 \tag{3.16}$$
$$u_3 = f_3(x_3)\sin\alpha x_1\sin\beta x_2,\quad p = f_4(x_3)\sin\alpha x_1\sin\beta x_2$$

Let us consider a rectangular plate $-a \leqslant x_1 \leqslant a,\ -b \leqslant x_2 \leqslant b$. We put $\alpha = m\pi/a,\ \beta = n\pi/b$ $(m, n = 0, 1, 2, \ldots)$. Then the solutions (3.16) in Problem B will satisfy the following boundary conditions at the side surface of the plate:

$$u_2 = 0;\quad u_3 = 0;\quad M_{11} = 0 \text{ when } x_1 = \pm a$$
$$u_1 = 0;\quad u_3 = 0;\quad M_{22} = 0 \text{ when } x_2 = \pm b$$
$$\left(M_{st} = \mathbf{M}_s\cdot\mathbf{i}_t;\quad \mathbf{M}_s = \pm\int_{-h}^{h}\mathbf{i}_s\cdot\mathbf{D}^{\cdot}x_3\,dx_3\right)$$

Consequently the solutions (3.16) will describe the flexural forms of the loss of stability in a hinged plate.

Let us compare the exact solution of the problem of stability with the results of an applied theorem of buckling of shells and plates based on the Kirchhoff hypotheses /9/. The analysis of the flexural forms of bifurcation of equilibrium in a compressed plate in the case when μ is an even function of the transverse coordinate x_3 is reduced, within the framework of this theory, to solving the following equation for the flexure of the middle surface $w(x_1, x_2)$ of the plate:

$$(\lambda^3 - 1)(\partial_1^2 w + \partial_2^2 w)\mu_1 + 2(\partial_1^4 w + 2\partial_1^2\partial_2^2 w + \partial_2^4 w)\mu_2 = 0 \tag{3.17}$$
$$\mu_1 = \int_0^h \mu(x_3)\,dx_3,\quad \mu_2 = \int_0^h \mu(x_3)\,x_3^2\,dx_3$$

Linearization of (3.17) relative to the critical deformation $e = 1 - \lambda$ yields, in the case of a homogeneous plate $(\mu = \text{const})$, the classical St. Venant equation /10/

$$1/3h^2\mu\nabla^4 w - \nabla\cdot\mathbf{T}\cdot\nabla w = 0$$

Writing w in the form $w = w_0\exp[i(\alpha x_1 + \beta x_2)]$ we obtain, from (3.17), the following equation for determining $\varkappa$:

$$\lambda^3 = 1 - 2\zeta^2\mu_2/\mu_1$$

and this yields

$$\lambda = 1 - \frac{2}{3}\zeta^2\mu_2/\mu_1 + O(\zeta^4) \tag{3.18}$$

Since $\mu(x_3)$ is an even function, the following expansion holds on the interval $[0, h]$:

$$\mu = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{\pi n x_3}{\varepsilon h} \tag{3.19}$$

From (3.19) we obtain

$$\mu_1 = a_0 h, \quad \mu_2 = a_0 h^3/3 + O(\varepsilon^2)$$

and in accordance with (3.18) we have

$$\lambda = 1 - \frac{2}{9} s^2 + O(s^4) \tag{3.20}$$

The solution of (3.13) has an asymptotic representation coinciding with (3.20) with the accuracy of up to the terms of order s^2. This means that the Kirchhoff theory gives correct results in the case of thin plates.

For the case $\nu = 2$ the solid lines in the figure depict the dependence of the critical deformation e on the relative thickness s of the plate. The curves *1* and *2* correspond to the problem A and B; curve *3* is obtained for the flexural forms of the plate equilibrium bifurcation using the Kirchhoff hypothesis. It is evident that in the case of thick plates the Kirchhoff theory leads to considerable errors.

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